Math 206B Lecture 3 Notes

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January 11, 2019

1 Irreducible Representations and Characters

1.1 Irreducible representations

Recall that if π, ρ are representations of G, then $\pi \oplus \rho$ is well-defined.

Definition 1.1. A representation ρ is **irreducible** if $\rho \neq \pi \oplus \pi'$.

Theorem 1.1. *Let G be a finite group. There exists a finite number of irreducible representations $\rho_0, \rho_1, \dots$ such that every $\pi = \bigoplus m_i \rho_i$ for some coefficients m_i. Moreover, $m_i = \langle \chi_\pi, \chi_{\rho_i} \rangle$.*

In other words, $\{\chi_{\rho_i}\}$ is an orthonormal basis in the space of functions $f : G \to \mathbb{C}$ that are constant on conjugacy classes.

Remark 1.1. The decomposition $\pi = \bigoplus m_i \rho_i$ is non-unique. G can act on $V = \mathbb{C}^d$ trivially. This representation is $\pi = \bigoplus d\rho_0$. Any basis will work. When all m_i are 0 or 1, this decomposition is in fact unique.

Example 1.1. Let π be the regular representation of G. Let χ_i be the character of an irreducible representation. Then $m_i = \langle \chi_\pi, \chi_i \rangle = |G|^{-1} \sum_{g \in G} \chi_\pi(g) \overline{\chi_i(g)} = |G|^{-1} \chi_\pi(1) \chi_i(1) = |G|^{-1} |G| \dim(\rho_i)$.

1.2 Irreducible characters and character tables

Let $\chi_0, \dots, \chi_{c-1}$ be the characters of the irreducible representations of G, where $c = c(G)$. Then $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$. Let $d_i = \chi_i(1) = \dim(\rho_i)$.

Theorem 1.2. $\sum_i d_i^2 = |G|$.

Theorem 1.3. $d_i \mid |G|$ *for all* i.

Theorem 1.4. *All χ_i are real iff $C^{-1} = C$ for all conjugacy classes C of G.*

Let's calculate character tables of irreducible representations of S_n.

Example 1.2.

S_2	id	$(1\,2)$
χ_0	1	1
χ_1	1	-1

For every n, there is the character $\chi_{\text{sign}}(\sigma) = (-1)^{\text{inv}(\sigma)}$, where $\text{inv}(\sigma)$ is the number of the inversions.

Proposition 1.1. $\text{inv}(\sigma\tau) \cong \text{inv}(\sigma) + \text{inv}(\tau) \pmod 2$..

Here is the character table for S_3:

Example 1.3.

S_3	id	$(1\,2)$	$(1\,2\,3)$
χ_0	1	1	1
χ_{sign}	1	-1	1
χ_1	2	0	-1

How do we find the values for χ_1? The first value is the dimension of the representation. Since $\chi_0 + \chi_{\text{sign}} + 2\chi_1 = \chi_\pi$, we can figure out the rest of the values.

What is the representation corresponding to the character χ_1? We can calculate $\langle \chi_0, \chi_{\text{nat}} \rangle = \dim(\rho^G)$ or, combinatorially, $= |G|^{-1} \sum_{\sigma \in S_n} \text{fixed pts}(\sigma) \cdot 1 = \frac{1}{n!} \sum_{i=1}^{n} \sum_{\sigma(i)=i} 1 = \frac{1}{n!} n(n-1)! = 1$. Then $\chi_1 = \chi_{\text{nat}} - \chi_0$.

Example 1.4.

S_4	id	$(1\,2)$	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$
χ_0	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_1	3	1	0	-1	-1
χ_1'	3	-1	0	-1	1
χ_2	2	0	-1	2	0

We can figure out χ_1 as the number of fixed points minus 1. Here, $\chi_1' = \chi_1 \cdot \chi_{\text{sign}}$. We can figure out χ_2 using the regular representation, as before.